VI. Options

- In the next sections we will study different types of options and their pricing
- Option is a special case of derivatives (contingent claims) which are the securities whose prices are determined by the prices of other securities
- Options allow a better risk management and make market more efficient
- In the following sections we will consider the two major types of options (also called vanilla options): calls and puts
- Call or put option is typically written on a stock share (underlying security)

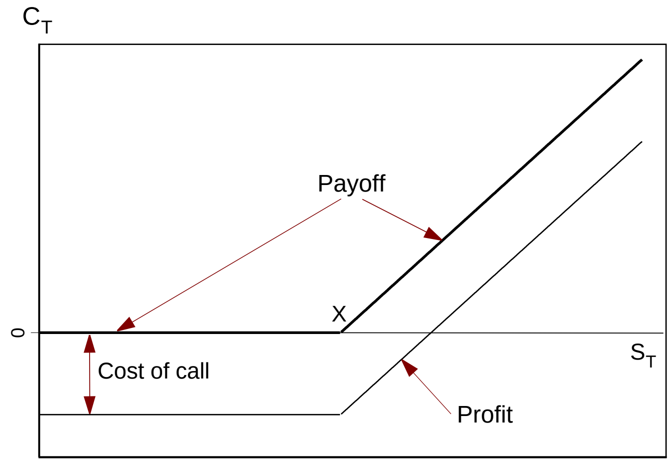
1. Definitions

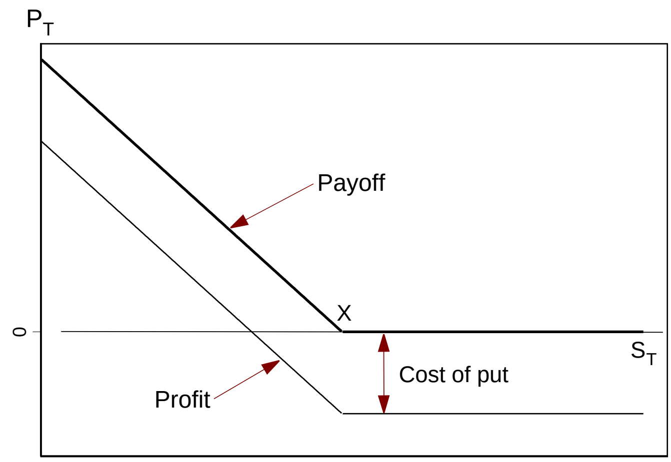
- A **call option** gives the holder the right to *purchase* a stock share for a specified price, called the *exercise* or *strike price*, on or before a given maturity date T.
- Holder of a call option will buy a stock share for the strike price X only if its price S_t at the exercise time t exceeds X.
- In particular, the value of the option C_T for its holder at expiration (also called the option payoff) is (also see the figure below)

$$C_T = (S_T - X)^+ \equiv \begin{cases} 0 & if\ \ S_T \leq X \\ S_T - X & if\ \ S_T > X. \end{cases}$$

- Investors who buy call options believe that the stock price will grow
- A **put option** gives the holder the right to *sell* a stock share for the exercise price, on or before maturity date T.
- The value of the option P_T at expiration is (also see the figure below)

$$P_T = (X - S_T)^+ \equiv \begin{cases} X - S_T & if\ \ S_T < X \\ 0 & if\ \ S_T \geq X. \end{cases}$$





- Investors who buy put options believe that the stock price will fall
- Profit on an option held to expiration is its value at expiration less the cost of purchase
- Call and Put options can be of two types: an **American option** and a **European option**
  - An American option allows its holder to exercise it before or at its expiration

  - A European option can be exercised only at its expiration
  - All options in Canada, except for stock index options are American options

- An option can be written on an individual stock, stock index, foreign currency, on bonds and other financial instruments
- The typical maturity date for an option is a few months

Practice Problem

Show the payoff and profit for a writer of a (a) call option, (b) put option

- Question: What are the expectations of a call (put) *seller* on the future of the underlying stock price?

- Each option is characterized by its moneyness
  - An option is *in–the–money* if its exercising is beneficial (apart from the cost of purchase)
    - It follows that a call option is in–the–money if $S > X$
    - A put option is in–the–money if $S < X$
  - An option is *out–of–the–money* if its exercising is not beneficial
    - It follows that a call option is out–of–the–money if $S < X$
    - A put option is out–of–the–money if $S > X$
  - An option is *at–the–money* if price of underlying stock is equal to the strike price ($S = X$)
  - Most of the options are issued at–the–money

2. Option Strategies

- Options could be used for speculations, taking advantage of mispricing in the markets and better risk management
- First, we consider the last application of options
- Due to market frictions such as transaction costs, illiquidity, margin requirements and so on, trading only bonds and stocks may not be sufficient to efficiently control the risk exposure of an investor

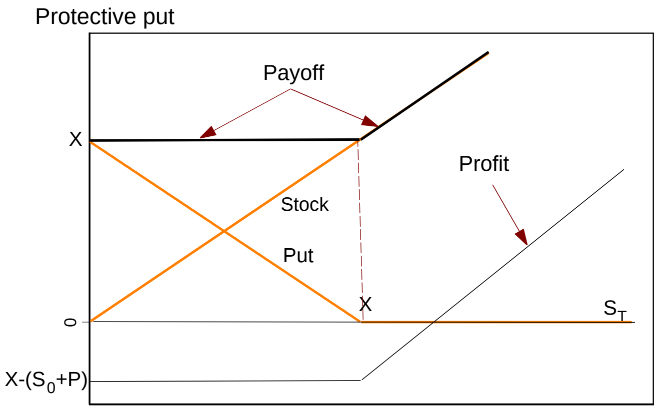
Protective Put

- If an investor bets on the stock growth but still wants to protect herself from downside risk then she buys a put while taking a long position in the underlying stock. This strategy is called *protective put.*

- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
Put	$\underline{X - S_T}$	$\underline{0}$
Total	X	S_T

- The graph of this payoff is shown below



- The cost of insurance against downside risk is the put price

- Typically, an investor must roll over her position in a put since its maturity is much shorter her investment horizon

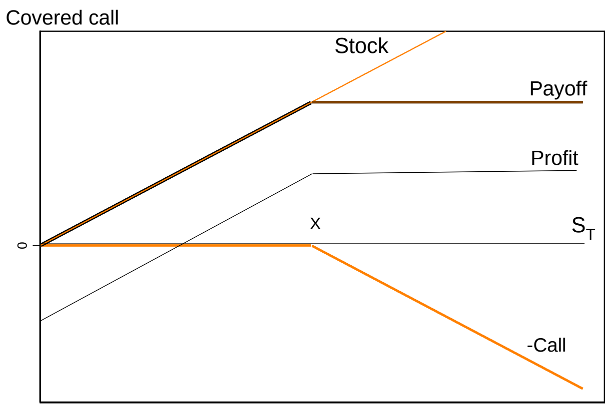
Covered Call

- A *covered call* position is the purchase of a share of stock with simultaneous sale of a call on that stock

- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
-Payoff of call	$\underline{0}$	$\underline{-(S_T - X)}$
Total	S_T	X

- The graph of this payoff is shown below



- This strategy is used by institutional investors who plan to sell the stock at price X

- These investors forfeit potential gains if stock rises above X but collect the premium for the call

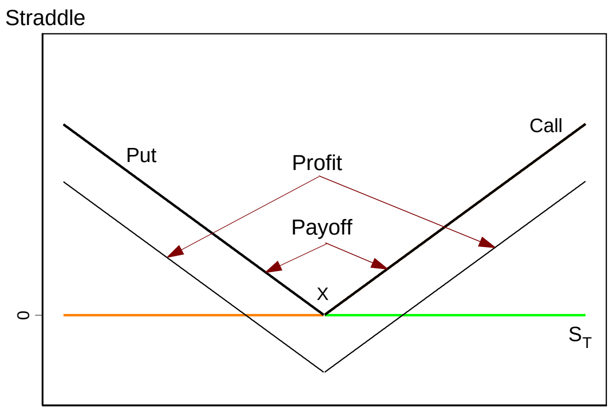
Straddle

- A long *straddle* is established by buying both a call and a put written on the same stock, each with the same exercise price, X, and the same expiration date, T.

- Payoff of this strategy is

	$S_T < X$	$S_T > X$
Call	0	$S_T - X$
Put	$X - S_T$	0
Total	$X - S_T$	$S_T - X$

- The graph of this payoff is shown below



- This strategy is used by investors who bet on volatility of a stock and uncertain about overall direction of the move

3. The Put–Call Parity

- Before discussing option pricing let us establish a simple relation between prices of put, call, underlying stock and riskless bond

- Recall that payoff of the protective put strategy is

	$S_T < X$	$S_T > X$
Stock	S_T	S_T
Put	$\underline{X - S_T}$	$\underline{0}$
Total	X	S_T

where T is a maturity date of a put option

- Now let us replicate the payoff of the protective put strategy by using T–bills and call options. Assume that call options are written on the same stock and have the same strike price and maturity as the put option does. T-bill pays one dollar at time T. The table below shows the number of calls and T–bills required to replicate protective put.

	$S_T < X$	$S_T > X$
______Call		
______T–bills	______	______
Total	X	S_T

- Assume that no arbitrage opportunities exist in the market
- Because the payoffs of the two strategies are the same, the costs to establish each portfolio must be equal
- Therefore

$$P_0 + S_0 = C + PV(X)$$

or

$$P_0 + S_0 = C + \frac{X}{1 + r_f},$$

where r_f is risk–free rate per time–period T

- The last equation is called the **put–call parity relationship**
- We emphasize that the put–call parity assumes that both call and put are written on the same stock, have the same expiration and strike price.

- The above put–call parity holds only if the stock does not pay dividends. With dividends, it becomes

$$P_0 + S_0 = C + \frac{X}{1 + r_f} + PV(D),$$

where $PV(D)$ is the present value of the dividends that will be paid by the stock during the life of the options.

Practice Problem

Suppose that the stock does not pay dividends and its current price is $100, the call price is $11, while the put price is $3. The expiration date of both options is in 6 months, their strike price is $100 and the annual effective risk–free interest rate is 10.25%. Check if the put–call parity is violated for these data. Do you have an access to an arbitrage opportunity here? If yes, then how do you trade to take advantage of it?

3. Option Pricing

- First, let us discuss the factors that affect option prices

Determinants of option values

The following table presents the effects of the relevant factors on call option prices

Increase in	Value of a Call Option
Stock price, S	Increases
Strike price, X	Decreases
Stock volatility, σ	Increases
Time to maturity, T	Increases
Interest rate, r_f	Increases
Dividends	Decreases

- Comments:
  - The higher the stock price today, the more likely that it will be high at the time of exercise
  - The higher X, the more likely that the difference $S - X$ will be lower at the time of exercise
  - Consider at–the–money call option with exercise price of \$50 written on the stock that could be equal to \$25 and \$75 with probabilities 0.5 at maturity date T compared with situation where the stock could be equal to \$40 and \$60 with probabilities 0.5. In both cases the expected stock price is \$50.

    The expected payoff of the option in the case of a more volatile stock is

    $$E(C_T) = 0.5 \times 0 + 0.5 \times (75 - 50) = \$12.5$$

    while the expected payoff of the option in the case of the other stock is

    $$E(C_T) = 0.5 \times 0 + 0.5 \times (60 - 50) = \$5$$

    * In summary, the price of a call option increases with higher volatility of stock because a stock fall has limited effect on the option payoff (C_T cannot be less than 0), while its rise increases the payoff proportionally.

- If T increases then volatility of stock increases too causing increase in the option price. Moreover, the present value of the strike price falls
- If r_f increases then the present value of the strike price falls causing increase in the option price.
- The stock price is a PV of its dividends paid during its life–time. The higher the stock price at option maturity, the higher the option price. Because the stock price today includes the PV of the dividends paid before option maturity and the stock value at maturity, the higher this PV the smaller the stock price would be at maturity date. Therefore, the option price falls with increasing PV of dividends paid before maturity

Practice Problem

Fill out the same table for a put option

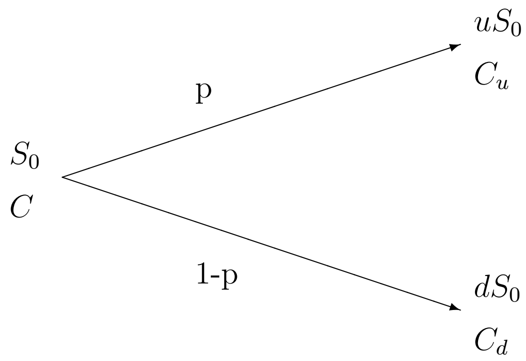
Increase in	Value of a Put Option
Stock price, S	
Strike price, X	
Stock return volatility, σ	
Time to maturity, T	
Interest rate, r_f	
Dividends	

Binomial Pricing of European Option

- We consider valuation of a European option written on a stock that does not pay dividends

- We can develop insight into option valuation by considering a simple special case when a stock can take only two possible states at option expiration and no states between today and option expiration. This model for the stock price is called single–period binomial model

- Assume that the stock price today is S_0 and it can increase at expiration date T to $S_u = uS_0$ (where $u > 1$) with probability p or fall to $S_d = dS_0$ (where $d < 1$) with probability $1 - p$



- Consider the call option that expires at T with an exercise price X. Suppose that the current price of a T-bill is B_0 and the riskless interest rate per time period T is r_f.

- The payoff of this option is shown below

	$S_T = dS_0$	$S_T = uS_0$
Payoff	$C_d = (dS_0 - X)^+$	$C_u = (uS_0 - X)^+$

- Suppose that an investor writes the option, sells it at the market price C and then use the proceeds to buy T-bills and stock shares. Let us exactly replicate the payoff of the call option by using replicating portfolio with H stock shares and Z riksless bonds. The payoff table for the replicating portfolio is shown below

	$S_T = dS_0$	$S_T = uS_0$
H shares of stock	$dS_0 \times H$	$uS_0 \times H$
Z T–bills	$ZB_0(1 + r_f)$	$ZB_0(1 + r_f)$
Total	C_d	C_u

- This table allows us to find a replicating portfolio, that is H and Z
- In particular, we must have

$$dS_0H + ZB_0(1 + r_f) = C_d$$

$$uS_0H + ZB_0(1 + r_f) = C_u$$

- We find the replicating portfolio after solving the last system of equations

$$H = \frac{C_u - C_d}{S_0(u - d)}$$

$$Z = \frac{C_d u - C_u d}{B_0(u - d)(1 + r_f)}$$

- We assume that the market does not allow arbitrage opportunities. Therefore, the price of the call option is equal to the value of the replicating portfolio:

$$C = HS_0 + ZB_0 = \frac{C_u - C_d}{(u - d)} + \frac{C_d u - C_u d}{(u - d)(1 + r_f)} = \frac{C_u(1 - d + r_f) + C_d(u - 1 - r_f)}{(u - d)(1 + r_f)}$$

- The last formula holds for any strike price. In the special case when $dS_0 < X < uS_0$ we find

$$C = \frac{C_u(1 - d + r_f) + C_d(u - 1 - r_f)}{(u - d)(1 + r_f)} = \frac{(uS_0 - X)(1 - d + r_f)}{(u - d)(1 + r_f)}$$

- Question: How the formula for the call price will change if the option is put?

- Number of the stock shares in the replicating portfolio, H, is called the hedge ratio.

- Question: What is the sign of H for a call (put) option?

- Note that the option price is independent from distribution of the stock returns (probability p)! Option price depends on the probability of the stock to go up (p) indirectly since the latter affects S_0

- It follows that the price of an option can be written as

$$C = E^Q(C_T)/(1+r_f) = (C_u P_u + C_d P_d)/(1+r_f),$$

where $P_u = (1-d+r_f)/[(u-d)]$ and $P_d = (u-1-r_f)/[(u-d)]$. P_u and P_d can be interpreted as probabilities of u and d–states, respectively, under a new probability measure Q. Probability measure Q is different than a real one (so p and P_u are not the same) and is called risk–neutral probability measure. It follows that price of an option is simply its expected payoff under risk–neutral measure discounted by a risk–free rate.

Example

Suppose that $S_0 = \$50$, $d = 0.8$, $u = 1.25$, $X = \$50$, $T = 0.25$, and the effective annual risk–free rate is $r_f = 5\%$. Find the price of the call option and the hedge ratio

The riskless rate of return for $T = 0.25$ is $(1+r_f)^{1/4} - 1 = 1.05^{1/4} - 1 = 1.227\%$. Given that $C_d = 0$ ($S_0 d = 50 \times 0.8 = 40 < X = 50$) and $C_u = S_0 u - X = 50 \times 1.25 - 50 = \$12.5$, we find

$$C = \frac{C_u(1-d+r_f) - C_d(1-u+r_f)}{(u-d)(1+r_f)} =$$

$$\frac{12.5(1-0.8+0.01227) - 0(1-1.25+0.01227)}{(1.25-0.8)(1+0.01227)} = \$5.82$$

The hedge ratio:

$$H = \frac{C_u - C_d}{S_0(u-d)} = \frac{12.5 - 0}{50(1.25-0.8)} = 0.56$$

It follows that an investor takes a long position in the stock to replicate the payoff of the call option. Moreover, her position in the replicating portfolio is highly leveraged: $\frac{HS}{C} = \frac{28}{5.82} = 4.81$

The dollar position in T-bills

$$ZB_0 = \frac{C_d u - C_u d}{(u-d)(1+r_f)} = C - H \times S_0 = 5.82 - 28 = -\$22.18$$

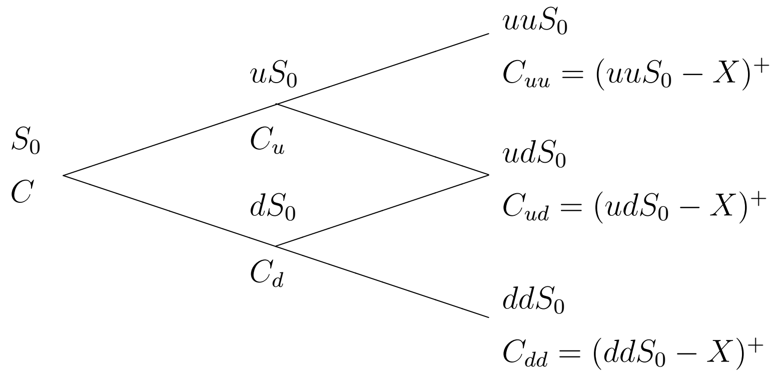
Question: What would be your arbitrage strategy if the call option were selling for \$3?

- Note that the interest rate usually has a negligible effect on an option price since the life of an option is relatively short

Practice Problem

Find the price of a put option in the example above assuming that it has the same expiration date and the exercise price as the call option and written on the same stock

- The two–state stock price model can be generalized to a multi–period model where the stock price can have multiple states at option expiration
- For example, consider the binomial tree for the two trading dates found by dividing the maturity date by 2
- The tree below shows the possible stock prices as well as the option prices
- The option price at each node of the tree is found by using backward induction:



- Consider the node where $S = uS_0$ and find the option price C_u by using the approach described above in which C_u, C_d, uS_0, and dS_0 are replaced by C_{uu}, C_{dd}, uuS_0, and udS_0. Use the interest rate per period in the formulas for C_u

- Repeat the calculations for the node where $S = dS_0$ to find the option price C_d

- Find the option price C for the node S_0 by using C_u and C_d found in the previous two steps

Practice Problem

Let r_f be a risk–free rate per period. Find the price of a call option, C, at the beginning of period 1 (see the diagram above) by using the risk–neutral probabilities, P_u, P_d defined on page 121.

Binomial Pricing of American Option

- We price American option by constructing binomial tree and using backward induction

- In each node of binomial tree, an investor has two choices: keep the option or exercise it. She makes the best decision implying that the value (price) of a call option in a given node n is

$$C_n^A = max\{(S_n - X)^+, C_n\},$$

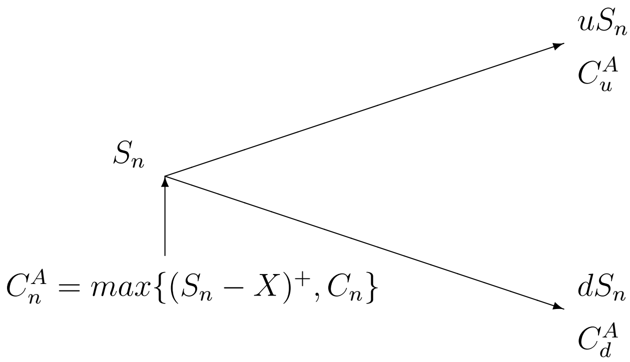
where S_n is the price of a stock at this node and C_n is the price of an option if it is not exercised given by

$$C_n = \frac{C_u^A(1 - d + r_f) + C_d^A(u - 1 - r_f)}{(u - d)(1 + r_f)},$$

where C_u^A, C_d^A are the prices of the American option in the following upper and lower nodes. See the diagram below

- If $(S_n - X)^+ > C_n$ then option should be exercised at node n. If $(S_n - X)^+ < C_n$ then option should not be exercised at node n

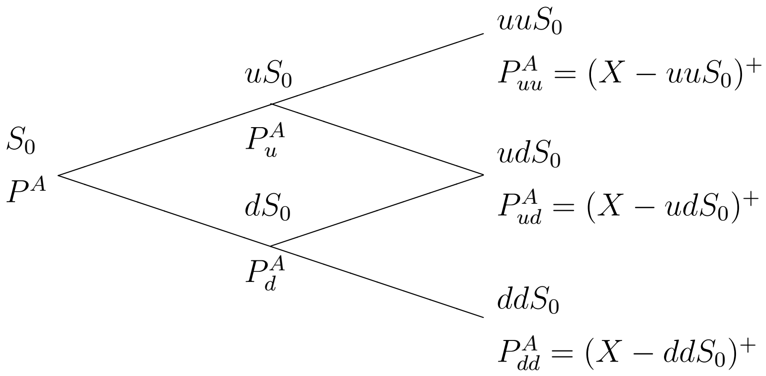
- The price of American put option is found the same way by using payoffs of the put



Example

Suppose that $S_0 = \$50$, $d = 0.75$, $u = 1.25$, $X = \$50$, $T = 0.5$, and the effective annual risk–free rate is $r_f = 5\%$. Find the price of the American put option today by using the two period binomial model.

Consider the diagram below.



First, we find $P^A_{uu} = (50 - 1.25^2 \times 50)^+ = 0$, $P^A_{ud} = (50 - 1.25 \times 0.75 \times 50)^+ =$ \$3.125, $P^A_{dd} = (50 - 0.75^2 \times 50)^+ =$ \$21.875

Let us look at the node uS_0. The stock price at this node is $uS_0 = 62.5$ and the value of option exercising is $(X - uS_0)^+ = 0$. If option is not exercised, then its value is

$$P_u = \frac{P^A_{uu}(1 - d + r_f) + P^A_{ud}(u - 1 - r_f)}{(u - d)(1 + r_f)} = \frac{3.125(1.25 - 1 - 0.01227)}{(1.25 - 0.75)(1 + 0.01227)} = \$1.47$$

It follows that the option price, P^A_u, at node uS_0 is \$1.47 and the option should not be exercised.

Now let us consider the node dS_0. If put is exercised then price is $(X - dS_0)^+ = 50 - .75 \times 50 = \$12.5$. If option is not exercised then its value is

$$P_d = \frac{P^A_{ud}(1 - d + r_f) + P^A_{dd}(u - 1 - r_f)}{(u - d)(1 + r_f)} =$$

$$\frac{3.125(1. - .75 + 0.01227) + 21.875(1.25 - 1 - 0.01227)}{(1.25 - 0.75)(1 + 0.01227)} = \$11.89.$$

It follows that the price, P^A_d, at node dS_0 is \$12.5 and the option should be exercised.

Finally, let us consider the node S_0. If option is exercised then the payoff is 0. If not then its value is

$$P = \frac{P^A_u(1 - d + r_f) + P^A_d(u - 1 - r_f)}{(u - d)(1 + r_f)} =$$

$$\frac{1.47(1. - .75 + 0.01227) + 12.5(1.25 - 1 - 0.01227)}{(1.25 - 0.75)(1 + 0.01227)} = \$6.63.$$

It follows that the no–arbitrage price today, P^A, should be \$6.63 and the option should not be exercised.

Additional Practice Problems

1. Buyers of put options anticipate the value of the underlying asset will and sellers of call options anticipate the value of the underlying asset will

A) increase; increase

B) decrease; increase

C) increase; decrease

D) decrease; decrease

E) cannot tell without further information

2. HighFlyer Stock currently sells for \$48. A one-year call option with strike price of \$55 sells for \$9, and the risk free interest rate is 6%. What is the price of a one-year put with strike price of \$55?

A) \$9.00

B) \$12.89

C) \$16.00

D) \$18.72

E) \$15.60

3. Use the two-state put option value in this problem. S_0 =100; X = 120; the two possibilities for ST are \$150 and \$80. The range of P across the two states is ; the hedge ratio is

A) \$0 and \$40; -4/7

B) \$0 and \$50; +4/7

C) \$0 and \$40; +4/7

D) \$0 and \$50; -4/7

E) \$20 and \$40; +1/2

4. A portfolio consists of 400 shares of stock and 200 calls on that stock. If the hedge ratio for the call is 0.6, what would be the dollar change in the value of the portfolio in response to a one dollar decline in the stock price?

A) +\$700

B) +\$500

C) -\$580

D) -\$520

E) none of the above

5. Suppose that $S_0 = \$50$, $d = 0.75$, $u = 1.25$, $X = \$50$, $T = 0.5$, and the effective annual risk–free rate is $r_f = 5\%$. Find the price of the American call option today by using the two period binomial model. Assume that the stock does not pay dividends.